Reduction and summation formulae for certain classes of generalised multiple hypergeometric series arising in physical and quantum chemical applications

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ADDENDUM

Reduction and summation formulae for certain classes of generalised multiple hypergeometric series arising in physical and quantum chemical applications

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Abstract. The multivariable hypergeometric function

$$F^{p_0:\,p_1;\,\ldots\,;\,p_n}_{q_0:\,q_1;\,\ldots\,;\,q_n}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

considered recently by Niukkanen and Srivastava, provides an interesting unification of the generalised hypergeometric function ${}_pF_q$ of one variable, Appell and Kampé de Fériet functions of two variables, and Lauricella functions of n variables, and also of many other hypergeometric series which arise naturally in various physical and quantum chemical applications. Indeed, as pointed out by Srivastava, the multivariable hypergeometric function is an obvious special case of the generalised Lauricella function of n variables, which was first introduced and studied by Srivastava and Daoust. By employing such connections of this multivariable hypergeometric function with the more general multiple hypergeometric functions studied in the literature rather systematically and widely, Srivastava presented several interesting and useful properties of this function, many of which were not given by Niukkanen. The object of this addendum to Srivastava's work is to derive a number of new reduction formulae for the multivariable hypergeometric function from substantially more general identities involving multiple series with essentially arbitrary terms. Some interesting summation formulae for the multivariable hypergeometric function with $x_1 = \ldots = x_n = 1$ and $x_1 = \ldots = x_n = -1$ are also presented.

1. Introduction and notations

For convenience, let

$$\boldsymbol{a} = (a^1, \ldots, a^p) \qquad \boldsymbol{b} = (b^1, \ldots, b^q) \tag{1}$$

and

$$\boldsymbol{a}_j = (a_j^1, \ldots, a_j^{p_j}) \qquad \boldsymbol{b}_j = (b_j^1, \ldots, b_j^{q_j}) \tag{2}$$

so that $\boldsymbol{a}$ and $\boldsymbol{b}$ are *vectors* with dimensions p and q, respectively, and

$$\boldsymbol{a}_j \text{ and } \boldsymbol{b}_j \qquad (j = 0, 1, \ldots, n)$$

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are *vectors* with dimensions p_j and q_j, respectively. Also, in terms of the Pochhammer symbol

$$(\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} = \begin{cases} 1 & \text{if } m=0 \\ \lambda(\lambda+1)\ldots(\lambda+m-1) & \text{if } m=1,2,3,\ldots \end{cases} \tag{3}$$

let

$$(\boldsymbol{a})_m = \prod_{k=1}^{p} (a^k)_m \qquad (\boldsymbol{b})_m = \prod_{k=1}^{q} (b^k)_m \tag{4}$$

and

$$(\boldsymbol{a}_j)_m = \prod_{k=1}^{p_j} (a_j^k)_m \qquad (\boldsymbol{b}_j)_m = \prod_{k=1}^{q_j} (b_j^k)_m \tag{5}$$

and define a generalised hypergeometric function of n variables by

$$F^{p_0:\,p_1;\,\ldots;\,p_n}_{q_0:\,q_1;\,\ldots;\,q_n}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \equiv F^{p_0:\,p_1;\,\ldots;\,p_n}_{q_0:\,q_1;\,\ldots;\,q_n}\left(\begin{matrix}\boldsymbol{a}_0:\,\boldsymbol{a}_1;\,\ldots;\,\boldsymbol{a}_n; \\ \boldsymbol{b}_0:\,\boldsymbol{b}_1;\,\ldots;\,\boldsymbol{b}_n;\end{matrix}\; x_1,\ldots,x_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\boldsymbol{a}_0)_{m_1+\ldots+m_n}}{(\boldsymbol{b}_0)_{m_1+\ldots+m_n}} \prod_{j=1}^{n} \left\{ \frac{(\boldsymbol{a}_j)_{m_j}}{(\boldsymbol{b}_j)_{m_j}} \frac{x_j^{m_j}}{m_j!} \right\} \tag{6}$$

where, for (absolute) convergence of the multiple hypergeometric series,

$$1+q_0+q_k-p_0-p_k \geqslant 0 \qquad (k=1,\ldots,n) \tag{7}$$

the equality holding true provided, in addition, we have *either*†

$$p_0 > q_0 \qquad \text{and} \qquad |x_1|^{1/(p_0-q_0)}+\ldots+|x_n|^{1/(p_0-q_0)} < 1 \tag{8}$$

or

$$p_0 \leqslant q_0 \qquad \text{and} \qquad \max\{|x_1|,\ldots,|x_n|\} < 1. \tag{9}$$

The works of Niukkanen (1983, 1984) and Srivastava (1985) on the multivariable hypergeometric function (6) are motivated by the large number of physical and quantum chemical applications of such multiple hypergeometric series (see, for numerous other applications, Exton (1976, chap 7 and 8; 1978, chap 7), Carlson (1977), Srivastava and Kashyap (1982) and Srivastava and Karlsson (1985, § 1.7)). Indeed, as already pointed out by Srivastava (1985), the multivariable hypergeometric function (6) is an obvious special case of the generalised Lauricella function of n variables, which was first introduced and studied by Srivastava and Daoust (1969, p 454 *et seq*), and this widely and systematically studied (Srivastava-Daoust) generalised Lauricella function has appeared in several subsequent works including, for example, two important books on the subject by Exton (1976, § 3.7, 1978, § 1.4), a book by Srivastava and Manocha (1984, p 64 *et seq*) and a book by Srivastava and Karlsson (1985, p 37 *et seq*); also, a *further* special case of the multivariable hypergeometric function (6) when

$$p_1=\ldots=p_n \qquad \text{and} \qquad q_1=\ldots=q_n \tag{10}$$

was considered earlier by Karlsson (1973). Srivastava (1985) employed these useful connections of (6) with the much more general multiple hypergeometric functions

† Under certain parametric constraints, the multiple hypergeometric series in (6) converges also when $x_k=\pm1$ $(k=1,\ldots,n)$, together with the equality in (7).

(studied in the literature rather systematically and widely) in order to present several interesting and useful properties of (6) (including, for example, regions of convergence, reduction and summation formulae, expansion and multiplication theorems, generating functions and operational formulae), many of which were not given by Niukkanen (1983, 1984). In this addendum to Srivastava (1985) we derive a number of new reduction formulae for the multivariable hypergeometric function (6) from substantially more general identities involving multiple series with essentially arbitrary terms. We also present some interesting summation formulae for (6) with

$$x_1 = \ldots = x_n = 1 \qquad \text{and} \qquad x_1 = \ldots = x_n = -1.$$

2. General series identities

Let $\{\Omega(n)\}_{n=0}^{\infty}$ be a *bounded* sequence of real (or complex) numbers, and set

$$L = l_1 + \ldots + l_n \qquad M = m_1 + \ldots + m_n \tag{11}$$

where l_j and m_j $(j = 1, \ldots, n)$ are non-negative integers. Then, from the works of Srivastava (1981), Buschman and Srivastava (1982) and Srivastava and Raina (1984), it is not difficult to establish the following general multiple-series identities:

$$\sum_{l_1, m_1, \ldots, l_n, m_n = 0}^{\infty} \Omega(L+M) \prod_{j=1}^{n} \left\{ (\lambda_j)_{l_j} (\mu_j)_{m_j} \frac{x_j^{l_j}}{l_j!} \frac{y_j^{m_j}}{m_j!} \right\}$$
$$= \sum_{l_1, m_1, \ldots, l_n, m_n = 0}^{\infty} \Omega(L+M) \prod_{j=1}^{n} \left\{ (\lambda_j)_{l_j} (\lambda_j + \mu_j + l_j)_{m_j} \frac{(x_j - y_j)^{l_j}}{l_j!} \frac{y_j^{m_j}}{m_j!} \right\}$$
$$\lambda_j + \mu_j \neq 0, -1, -2, \ldots \qquad (j = 1, \ldots, n) \tag{12}$$

$$\sum_{l_1, m_1, \ldots, l_n, m_n = 0}^{\infty} \frac{\Omega(L+M)}{\prod_{j=1}^{n} \{(\rho_j)_{l_j} (\sigma_j)_{m_j}\}} \prod_{j=1}^{n} \left\{ \frac{x_j^{l_j}}{l_j!} \frac{y_j^{m_j}}{m_j!} \right\}$$
$$= \sum_{l_1, m_1, \ldots, l_n, m_n = 0}^{\infty} \Omega(L+M) \prod_{j=1}^{n} \left\{ \frac{(\rho_j + \sigma_j + l_j + m_j - 1)_{m_j}}{(\rho_j)_{l_j + m_j} (\sigma_j)_{m_j}} \frac{(x_j - y_j)^{l_j}}{l_j!} \frac{y_j^{m_j}}{m_j!} \right\}$$
$$\rho_j, \sigma_j \neq 0, -1, -2, \ldots \qquad (j = 1, \ldots, n) \tag{13}$$

$$\sum_{l_1, m_1, \ldots, l_n, m_n = 0}^{\infty} \Omega(L+M) \prod_{j=1}^{n} \left\{ \frac{(\lambda_j)_{l_j} (\lambda_j)_{m_j}}{(\rho_j)_{l_j} (\rho_j)_{m_j}} \frac{x_j^{l_j}}{l_j!} \frac{y_j^{m_j}}{m_j!} \right\}$$
$$= \sum_{l_1, m_1, \ldots, l_n, m_n = 0}^{\infty} \Omega(L+2M) \prod_{j=1}^{n} \left\{ \frac{(\lambda_j)_{l_j + m_j} (\rho_j - \lambda_j)_{m_j}}{(\rho_j)_{l_j + 2m_j} (\rho_j)_{m_j}} \frac{(x_j + y_j)^{l_j}}{l_j!} \frac{(-x_j y_j)^{m_j}}{m_j!} \right\}$$
$$\rho_j \neq 0, -1, -2, \ldots \qquad (j = 1, \ldots, n) \tag{14}$$

$$\sum_{l_1, m_1, \ldots, l_n, m_n = 0}^{\infty} \Omega(L+M) \prod_{j=1}^{n} \left\{ (\lambda_j)_{l_j} (\lambda_j)_{m_j} (\mu_j)_{l_j} (\mu_j)_{m_j} \frac{x_j^{l_j}}{l_j!} \frac{y_j^{m_j}}{m_j!} \right\}$$
$$= \sum_{l_1, m_1, \ldots, l_n, m_n = 0}^{\infty} \Omega(L+2M)$$
$$\times \prod_{j=1}^{n} \left\{ \frac{(\lambda_j + \mu_j)_{l_j + 2m_j} (\lambda_j)_{l_j + m_j} (\mu_j)_{l_j + m_j}}{(\lambda_j + \mu_j)_{l_j + m_j}} \frac{(x_j + y_j)^{l_j}}{l_j!} \frac{(-x_j y_j)^{m_j}}{m_j!} \right\}$$
$$\lambda_j + \mu_j \neq 0, -1, -2, \ldots \qquad (j = 1, \ldots, n) \tag{15}$$

$$\sum_{l_1,m_1,\ldots,l_n,m_n=0}^{\infty} \frac{\Omega(L+M)}{\prod_{j=1}^{n}\{(\rho_j)_{l_j}(\rho_j)_{m_j}(\sigma_j)_{l_j}(\sigma_j)_{m_j}\}} \prod_{j=1}^{n}\left\{\frac{x_j^{l_j}}{l_j!}\frac{y_j^{m_j}}{m_j!}\right\}$$

$$= \sum_{l_1,m_1,\ldots,l_n,m_n=0}^{\infty} \Omega(L+2M)$$

$$\times \prod_{j=1}^{n}\left\{\frac{(\rho_j+\sigma_j+l_j+2m_j-1)_{m_j}}{(\rho_j)_{l_j+2m_j}(\sigma_j)_{l_j+2m_j}(\rho_j)_{m_j}(\sigma_j)_{m_j}} \frac{(x_j+y_j)^{l_j}}{l_j!}\frac{(x_jy_j)^{m_j}}{m_j!}\right\}$$

$$\rho_j, \sigma_j, \rho_j+\sigma_j-1 \neq 0, -1, -2, \ldots \qquad (j=1,\ldots,n) \tag{16}$$

$$\sum_{l_1 m_1,\ldots,l_n,m_n=0}^{\infty} \Omega(L+M) \prod_{j=1}^{n}\left\{(\lambda_j)_{l_j}(\lambda_j)_{m_j}\frac{x_j^{l_j}}{l_j!}\frac{y_j^{m_j}}{m_j!}\right\}$$

$$= \sum_{l_1,m_1,\ldots,l_n,m_n=0}^{\infty} \Omega(L+2M) \prod_{j=1}^{n}\left\{(\lambda_j)_{l_j+m_j}\frac{(x_j+y_j)^{l_j}}{l_j!}\frac{(-x_jy_j)^{m_j}}{m_j!}\right\} \tag{17}$$

and

$$\sum_{l_1,m_1,\ldots,l_n,m_n=0}^{\infty} \frac{\Omega(L+M)}{\prod_{j=1}^{n}\{(\rho_j)_{l_j}(\rho_j)_{m_j}\}} \prod_{j=1}^{n}\left\{\frac{x_j^{l_j}}{l_j!}\frac{y_j^{m_j}}{m_j!}\right\}$$

$$= \sum_{l_1,m_1,\ldots,l_n,m_n=0}^{\infty} \frac{\Omega(L+2M)}{\prod_{j=1}^{n}\{(\rho_j)_{l_j+2m_j}(\rho_j)_{m_j}\}} \prod_{j=1}^{n}\left\{\frac{(x_j+y_j)^{l_j}}{l_j!}\frac{(x_jy_j)^{m_j}}{m_j!}\right\}$$

$$\rho_j \neq 0, -1, -2, \ldots \qquad (j=1,\ldots,n) \tag{18}$$

provided that the multiple series involved in each identity are absolutely convergent.

In view of the principle of confluence exhibited by

$$\lim_{\lambda\to\infty}\left\{(\lambda)_m\left(\frac{z}{\lambda}\right)^m\right\} = z^m = \lim_{\mu\to\infty}\left\{\frac{(\mu z)^m}{(\mu)_m}\right\} \tag{19}$$

for bounded z and $m=0,1,2,\ldots$, the series identity (17) would follow readily from (15) if in (15) we first replace x_j by x_j/μ_j and y_j by y_j/μ_j, and then let $\mu_j\to\infty$ $(j=1,\ldots,n)$. Indeed, the series identity (17) also follows from (14) if in (14) we first replace x_j by $\rho_j x_j$ and y_j by $\rho_j y_j$, and then let $\rho_j\to\infty$ $(j=1,\ldots,n)$. Formula (18), on the other hand, follows similarly from (14) as well as (16).

3. Applications to multiple hypergeometric series

By setting $y_j=x_j$ $(j=1,\ldots,n)$ in the multiple-series identities (12) and (13), and $y_j=-x_j$ $(j=1,\ldots,n)$ in (14)-(18), we shall immediately obtain various reduction formulae for multiple series with essentially arbitrary terms. As a matter of fact, a remarkably large number of similar multiple-series reduction formulae can be derived directly from the works of Srivastava (1973, 1981), Buschman and Srivastava (1982), Karlsson (1982, 1983, 1984) and Srivastava and Karlsson (1985, §§ 1.3 and 1.4). In each of these reduction formulae involving multiple series with essentially arbitrary terms, if we *further* set (cf (3) and (4))

$$\Omega(n) = (\boldsymbol{a})_n/(\boldsymbol{b})_n \qquad (n=0,1,2,\ldots) \tag{20}$$

we shall readily arrive at the corresponding reduction formula for a multivariable hypergeometric function defined by (6). Some of the hypergeometric reduction formulae thus obtained are listed below:

$$F^{p:1;\dots;1}_{q:0;\dots;0}\left(\begin{matrix}\mathbf{a}: \lambda_1;\dots;\lambda_n;\mu_1;\dots;\mu_n;\\ \mathbf{b}: \text{—};\dots;\text{—};\text{—};\dots;\text{—};\end{matrix}\; x_1,\dots,x_n,x_1,\dots,x_n\right)$$

$$= F^{p:1;\dots;1}_{q:0;\dots;0}\left(\begin{matrix}\mathbf{a}: \lambda_1+\mu_1;\dots;\lambda_n+\mu_n;\\ \mathbf{b}: \text{———};\dots;\text{———};\end{matrix}\; x_1,\dots,x_n\right) \qquad (21)$$

$$F^{p:0;\dots;0}_{q:1;\dots;1}\left(\begin{matrix}\mathbf{a}: \text{—};\dots;\text{—};\text{—};\dots;\text{—};\\ \mathbf{b}: \rho_1;\dots;\rho_n;\sigma_1;\dots;\sigma_n;\end{matrix}\; x_1,\dots,x_n,x_1,\dots,x_n\right)$$

$$= F^{p:2;\dots;2}_{q:3;\dots;3}\left(\begin{matrix}\mathbf{a}: \Delta(2;\rho_1+\sigma_1-1);\dots;\Delta(2;\rho_n+\sigma_n-1);\\ \mathbf{b}: \rho_1,\sigma_1,\rho_1+\sigma_1-1;\dots;\rho_n,\sigma_n,\rho_n+\sigma_n-1;\end{matrix}\; 4x_1,\dots,4x_n\right) \qquad (22)$$

where, and in what follows, $\Delta(l;\lambda)$ abbreviates the array of l parameters:

$$\frac{\lambda}{l},\frac{\lambda+1}{l},\dots,\frac{\lambda+l-1}{l} \qquad (l=1,2,3,\dots)$$

$$F^{p:1;\dots;1}_{q:1;\dots;1}\left(\begin{matrix}\mathbf{a}: \lambda_1;\dots;\lambda_n;\lambda_1;\dots;\lambda_n;\\ \mathbf{b}: \rho_1;\dots;\rho_n;\rho_1;\dots;\rho_n;\end{matrix}\; x_1,\dots,x_n,-x_1,\dots,-x_n\right)$$

$$= F^{2p:2;\dots;2}_{2q:3;\dots;3}\left(\begin{matrix}\Delta(2;\mathbf{a}): \lambda_1,\;\rho_1-\lambda_1;\dots;\lambda_n,\;\rho_n-\lambda_n;\\ \Delta(2;\mathbf{b}): \rho_1,\Delta(2;\rho_1);\dots;\rho_n,\Delta(2;\rho_n);\end{matrix}\; 4^{p-q-1}x_1^2,\dots,4^{p-q-1}x_n^2\right) \qquad (23)$$

$$F^{p:2;\dots;2}_{q:0;\dots;0}\left(\begin{matrix}\mathbf{a}: \lambda_1,\mu_1;\dots;\lambda_n,\mu_n;\lambda_1,\mu_1;\dots;\lambda_n,\mu_n;\\ \mathbf{b}: \text{———};\dots;\text{———};\text{———};\dots;\text{———};\end{matrix}\; x_1,\dots,x_n,-x_1,\dots,-x_n\right)$$

$$= F^{2p:4;\dots;4}_{2q:1;\dots;1}\left(\begin{matrix}\Delta(2;\mathbf{a}): \lambda_1,\mu_1,\Delta(2;\lambda_1+\mu_1);\dots;\lambda_n,\mu_n,\Delta(2;\lambda_n+\mu_n);\\ \Delta(2;\mathbf{b}): \lambda_1+\mu_1;\dots;\lambda_n+\mu_n;\end{matrix}\; 4^{p-q+1}x_1^2,\dots,4^{p-q+1}x_n^2\right) \qquad (24)$$

$$F^{p:0;\dots;0}_{q:2;\dots;2}\left(\begin{matrix}\mathbf{a}: \text{———};\dots;\text{———};\text{———};\dots;\text{———};\\ \mathbf{b}: \rho_1,\sigma_1;\dots;\rho_n,\sigma_n;\rho_1,\sigma_1;\dots;\rho_n,\sigma_n;\end{matrix}\; x_1,\dots,x_n,-x_1,\dots,-x_n\right) =$$

$$F^{2p:3;\dots;3}_{2q:8;\dots;8}\left(\begin{matrix}\Delta(2;\mathbf{a}): \Delta(3;\rho_1+\sigma_1-1);\dots;\Delta(3;\rho_n+\sigma_n-1);\\ \Delta(2;\mathbf{b}): \rho_1,\sigma_1,\Delta(2;\rho_1),\Delta(2;\sigma_1),\Delta(2;\rho_1+\sigma_1-1);\dots;\rho_n,\sigma_n,\Delta(2;\rho_n),\Delta(2;\sigma_n),\Delta(2;\rho_n+\sigma_n-1);\end{matrix}\; \zeta_1,\dots,\zeta_n\right) \qquad (25)$$

where, for convenience,

$$\zeta_j = -4^{p-q-3}27x_j^2 \qquad (j=1,\dots,n)$$

$$F^{p:1;\dots;1}_{q:0;\dots;0}\left(\begin{matrix}\mathbf{a}: \lambda_1;\dots;\lambda_n;\lambda_1;\dots;\lambda_n;\\ \mathbf{b}: \text{—};\dots;\text{—};\text{—};\dots;\text{—};\end{matrix}\; x_1,\dots,x_n,-x_1,\dots,-x_n\right)$$

$$= F^{2p:1;\dots;1}_{2q:0;\dots;0}\left(\begin{matrix}\Delta(2;\mathbf{a}): \lambda_1;\dots;\lambda_n;\\ \Delta(2;\mathbf{b}): \text{—};\dots;\text{—};\end{matrix}\; 4^{p-q}x_1^2,\dots,4^{p-q}x_n^2\right) \qquad (26)$$

$$F^{p:0;\dots;0}_{q:1;\dots;1}\left(\begin{matrix}\mathbf{a}: \text{—};\dots;\text{—};\text{—};\dots;\text{—};\\ \mathbf{b}: \rho_1;\dots;\rho_n;\rho_1;\dots;\rho_n;\end{matrix}\; x_1,\dots,x_n,-x_1,\dots,-x_n\right)$$

$$= F^{2p:0;\dots;0}_{2q:3;\dots;3}\left(\begin{matrix}\Delta(2;\mathbf{a}): \text{————};\dots;\text{————};\\ \Delta(2;\mathbf{b}): \rho_1,\Delta(2;\rho_1);\dots;\rho_n,\Delta(2;\rho_n);\end{matrix}\; -4^{p-q-1}x_1^2,\dots,-4^{p-q-1}x_n^2\right) \qquad (27)$$

$$F^{p:1;\dots;1}_{q:1;\dots;1}\left(\begin{matrix}\mathbf{a}: \lambda_1;\dots;\lambda_n;\mu_1;\dots;\mu_n;\\ \mathbf{b}: 2\lambda_1;\dots;2\lambda_n;2\mu_1;\dots;2\mu_n;\end{matrix}\; x_1,\dots,x_n,-x_1,\dots,-x_n\right) =$$

$$F^{2p:2;\dots;2}_{2q:3;\dots;3}\left(\begin{matrix}\Delta(2;\mathbf{a}): \Delta(2;\lambda_1+\mu_1);\dots;\\ \Delta(2;\mathbf{b}): \lambda_1+1/2,\mu_1+1/2,\lambda_1+\mu_1;\dots;\end{matrix}\right.$$

$$\left.\begin{matrix}\Delta(2;\lambda_n+\mu_n);\\ \lambda_n+1/2,\mu_n+1/2,\lambda_n+\mu_n;\end{matrix}\; 4^{p-q-1}x_1^2,\dots,4^{p-q-1}x_n^2\right) \qquad (28)$$

$$F^{p:2;\dots;2}_{q:1;\dots;1}\left(\begin{matrix}\mathbf{a}: \lambda_1,\lambda_1+1/2;\dots;\lambda_n,\lambda_n+1/2;\mu_1,\mu_1+1/2;\dots;\mu_n,\mu_n+1/2;\\ \mathbf{b}: 2\lambda_1+1;\dots;2\lambda_n+1;2\mu_1+1;\dots;2\mu_n+1;\end{matrix}\; x_1,\dots,x_n,x_1,\dots,x_n\right)$$

$$= F^{p:2;\dots;2}_{q:1;\dots;1}\left(\begin{matrix}\mathbf{a}: \lambda_1+\mu_1,\lambda_1+\mu_1+1/2;\dots;\lambda_n+\mu_n,\lambda_n+\mu_n+1/2;\\ \mathbf{b}: 2\lambda_1+2\mu_1+1;\dots;2\lambda_n+2\mu_n+1;\end{matrix}\; x_1,\dots,x_n\right) \qquad (29)$$

$$F^{p:2;\dots;2}_{q:1;\dots;1}\left(\begin{matrix}\mathbf{a}: \lambda_1,\mu_1;\dots;\lambda_n,\mu_n;\lambda_1,\mu_1;\dots;\lambda_n,\mu_n;\\ \mathbf{b}: \lambda_1+\mu_1+1/2;\dots;\lambda_n+\mu_n+1/2;\lambda_1+\mu_1+1/2;\dots;\lambda_n+\mu_n+1/2;\end{matrix}\; x_1,\dots,x_n,x_1,\dots,x_n\right)$$

$$= F^{p:3;\dots;3}_{q:2;\dots;2}\left(\begin{matrix}\mathbf{a}: 2\lambda_1,2\mu_1,\lambda_1+\mu_1;\dots;2\lambda_n,2\mu_n,\lambda_n+\mu_n;\\ \mathbf{b}: \lambda_1+\mu_1+1/2,2\lambda_1+2\mu_1;\dots;\lambda_n+\mu_n+1/2,2\lambda_n+2\mu_n;\end{matrix}\; x_1,\dots,x_n\right) \qquad (30)$$

$$F^{p:2;\dots;2}_{q:1;\dots;1}\left(\begin{matrix}\mathbf{a}: \lambda_1,\mu_1;\dots;\lambda_n,\mu_n;\lambda_1,\mu_1;\dots;\lambda_n,\mu_n;\\ \mathbf{b}: \lambda_1+\mu_1-1/2;\dots;\lambda_n+\mu_n-1/2;\lambda_1+\mu_1+1/2;\dots;\lambda_n+\mu_n+1/2;\end{matrix}\; x_1,\dots,x_n,x_1,\dots,x_n\right)$$

$$= F^{p:3;\dots;3}_{q:2;\dots;2}\left(\begin{matrix}\mathbf{a}: 2\lambda_1,2\mu_1,\lambda_1+\mu_1;\dots;2\lambda_n,2\mu_n,\lambda_n+\mu_n;\\ \mathbf{b}: \lambda_1+\mu_1+1/2,2\lambda_1+2\mu_1-1;\dots;\lambda_n+\mu_n+1/2,2\lambda_n+2\mu_n-1;\end{matrix}\; x_1,\dots,x_n\right) \qquad (31)$$

$$F^{p:2;\dots;2}_{q:1;\dots;1}\left(\begin{matrix}\mathbf{a}: \lambda_1,\mu_1;\dots;\lambda_n,\mu_n;\lambda_1\;\mu_1-1;\dots;\lambda_n,\mu_n-1;\\ \mathbf{b}: \lambda_1+\mu_1-1/2;\dots;\lambda_n+\mu_n-1/2;\lambda_1+\mu_1-1/2;\dots;\lambda_n+\mu_n-1/2;\end{matrix}\; x_1,\dots,x_n,x_1,\dots,x_n\right)$$

$$= F^{p:3;\dots;3}_{q:2;\dots;2}\left(\begin{matrix}\mathbf{a}: 2\lambda_1,2\mu_1-1,\lambda_1+\mu_1-1;\dots;2\lambda_n,2\mu_n-1,\lambda_n+\mu_n-1;\\ \mathbf{b}: \lambda_1+\mu_1-1/2,2\lambda_1+2\mu_1-2;\dots;\lambda_n+\mu_n-1/2,2\lambda_n+2\mu_n-2;\end{matrix}\; x_1,\dots,x_n\right) \qquad (32)$$

$$F^{p:2;\dots;2}_{q:1;\dots;1}\left(\begin{matrix}\mathbf{a}: \lambda_1,\mu_1;\dots;\lambda_n,\mu_n;1/2-\lambda_1,1/2-\mu_1;\dots;1/2-\lambda_n,1/2-\mu_n;\\ \mathbf{b}: \lambda_1+\mu_1+1/2;\dots;\lambda_n+\mu_n+1/2;3/2-\lambda_1-\mu_1;\dots;3/2-\lambda_n-\mu_n;\end{matrix}\; x_1,\dots,x_n,x_1,\dots,x_n\right)$$

$$= F^{p:3;\dots;3}_{q:2;\dots;2}\left(\begin{matrix}\mathbf{a}: \lambda_1-\mu_1+1/2,\mu_1-\lambda_1+1/2,1/2;\dots;\lambda_n-\mu_n+1/2,\mu_n-\lambda_n+1/2,1/2;\\ \mathbf{b}: \lambda_1+\mu_1+1/2,3/2-\lambda_1-\mu_1;\dots;\lambda_n+\mu_n+1/2,3/2-\lambda_n-\mu_n;\end{matrix}\; x_1,\dots,x_n\right) \qquad (33)$$

$$F^{p:\,2;\,\ldots;\,2}_{q:\,1;\,\ldots;\,1}\left(\begin{matrix}\mathbf{a}:\ \lambda_1-1/2,\mu_1-1/2;\ldots;\lambda_n-1/2,\mu_n-1/2;\ \lambda_1+1/2,\mu_1+1/2;\ldots;\lambda_n+1/2,\mu_n+1/2;\\ \mathbf{b}:\ \lambda_1+\mu_1-1/2;\ldots;\lambda_n+\mu_n-1/2;\ \lambda_1+\mu_1+1/2;\ldots;\lambda_n+\mu_n+1/2;\end{matrix}\ x_1,\ldots,x_n,x_1,\ldots,x_n\right)$$

$$=F^{p:\,3;\,\ldots;\,3}_{q:\,2;\,\ldots;\,2}\left(\begin{matrix}\mathbf{a}:\ 2\lambda_1,2\mu_1,\lambda_1+\mu_1;\ldots;2\lambda_n,2\mu_n,\lambda_n+\mu_n;\\ \mathbf{b}:\ \lambda_1+\mu_1+1/2,2\lambda_1+2\mu_1-1;\ldots;\lambda_n+\mu_n+1/2,2\lambda_n+2\mu_n-1;\end{matrix}\ x_1,\ldots,x_n\right) \tag{34}$$

$$F^{p:\,2;\,\ldots;\,2}_{q:\,1;\,\ldots;\,1}\left(\begin{matrix}\mathbf{a}:\ \lambda_1+1/2,\mu_1-1/2;\ldots;\lambda_n+1/2,\mu_n-1/2;\ \lambda_1+1/2,\mu_1+1/2;\ldots;\lambda_n+1/2,\mu_n+1/2;\\ \mathbf{b}:\ \lambda_1+\mu_1+1/2;\ldots;\lambda_n+\mu_n+1/2;\ \lambda_1+\mu_1+1/2;\ldots;\lambda_n+\mu_n+1/2;\end{matrix}\ x_1,\ldots,x_n,x_1,\ldots,x_n\right)$$

$$=F^{p:\,3;\,\ldots;\,3}_{q:\,2;\,\ldots;\,2}\left(\begin{matrix}\mathbf{a}:\ 2\lambda_1+1,2\mu_1,\lambda_1+\mu_1;\ldots;2\lambda_n+1,2\mu_n,\lambda_n+\mu_n;\\ \mathbf{b}:\ \lambda_1+\mu_1+1/2,2\lambda_1+2\mu_1;\ldots;\lambda_n+\mu_n+1/2,2\lambda_n+2\mu_n;\end{matrix}\ x_1,\ldots,x_n\right) \tag{35}$$

$$F^{p:\,2;\,\ldots;\,2}_{q:\,1;\,\ldots;\,1}\left(\begin{matrix}\mathbf{a}:\ \lambda_1,\lambda_1+1/2;\ldots;\lambda_n,\lambda_n+1/2;\ 1-\lambda_1,3/2-\lambda_1;\ldots;1-\lambda_n,3/2-\lambda_n;\\ \mathbf{b}:\ 2\lambda_1;\ldots;2\lambda_n;\ 2-2\lambda_1;\ldots;2-2\lambda_n;\end{matrix}\ x_1,\ldots,x_n,x_1,\ldots,x_n\right)$$

$$=F^{p:\,1;\,\ldots;\,1}_{q:\,0;\,\ldots;\,0}\left(\begin{matrix}\mathbf{a}:\ 1;\ldots;1;\\ \mathbf{b}:\ -;\ldots;-;\end{matrix}\ x_1,\ldots,x_n\right). \tag{36}$$

Here, as is quite usual in the theory of hypergeometric functions, an empty set of parameters is represented by a dash, and the reduction formulae (21)-(36) hold true whenever the multiple hypergeometric functions involved in each formula exist.

By appealing to one or the other known summation theorems for generalised hypergeometric series (see Slater 1966, Luke 1975), we can derive a fairly large number of interesting summation formulae for multiple hypergeometric series of the type (6) as the consequence of the reduction formulae (21)-(36) and those listed by Srivastava (1985, L229). For example, in view of certain familiar summation theorems for the one-variable hypergeometric series ${}_7F_6$, ${}_6F_5$ and ${}_2F_1$, we readily obtain the following summation formulae for multiple hypergeometric series (cf Srivastava 1984):

$$F^{6:\,1;\,\ldots;\,1}_{6:\,0;\,\ldots;\,0}\left(\begin{matrix}\alpha,1+\alpha/2,\beta,\gamma,\delta,\varepsilon:\ -s_1;\ldots;-s_n;\\ \alpha/2,1+\alpha-\beta,1+\alpha-\gamma,1+\alpha-\delta,1+\alpha-\varepsilon,1+\alpha-S:\ -;\ldots;-;\end{matrix}\ 1,\ldots,1\right)$$

$$=\frac{(1+\alpha)_S(1+\alpha-\beta-\gamma)_S(1+\alpha-\beta-\delta)_S(1+\alpha-\gamma-\delta)_S}{(1+\alpha-\beta)_S(1+\alpha-\gamma)_S(1+\alpha-\delta)_S(1+\alpha-\beta-\gamma-\delta)_S} \tag{37}$$

where $s_1,\ldots,s_n$ are non-negative integers, and

$$1+2\alpha=\beta+\gamma+\delta+\varepsilon-S \qquad S\equiv s_1+\ldots+s_n$$

$$F^{5:\,1;\,\ldots;\,1}_{5:\,0;\,\ldots;\,0}\left(\begin{matrix}\alpha,1+\alpha/2,\beta,\gamma,\delta:\ \lambda_1;\ldots;\lambda_n;\\ \alpha/2,1+\alpha-\beta,1+\alpha-\gamma,1+\alpha-\delta,1+\alpha-\Lambda:\ -;\ldots;-;\end{matrix}\ -1,\ldots,-1\right)$$

$$=\frac{\Gamma(1+\alpha-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\alpha-\delta)\Gamma(1+\alpha-\Lambda)}{\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+\alpha-\beta-\delta)\Gamma(\alpha+\beta+\delta)}$$
$$\times\frac{\Gamma(1+\frac{1}{2}+\frac{1}{2}\alpha-\frac{1}{2}\gamma-\frac{1}{2}\Lambda)\Gamma(\frac{1}{2}+\frac{1}{2}\alpha-\frac{1}{2}\beta-\frac{1}{2}\delta)}{\Gamma(1+\frac{1}{2}\alpha-\frac{1}{2}\beta-\frac{1}{2}\gamma)\Gamma(1+\frac{1}{2}\alpha-\frac{1}{2}\delta-\frac{1}{2}\Lambda)} \tag{38}$$

where

$$\mathrm{Re}(\alpha)>0 \qquad \beta+\Lambda=\gamma+\delta=1 \qquad \Lambda\equiv\lambda_1+\ldots+\lambda_n$$

and (cf Karlsson 1983)

$$F^{1:\,2;\,\ldots;\,2}_{1:\,1;\,\ldots;\,1}\left(\begin{matrix}\alpha:\ \lambda_1,2\lambda_1+1;\ldots;\lambda_n,2\lambda_n+1;\\ 1/2+\alpha/2+n/2+\Lambda:\ 2\lambda_1;\ldots;2\lambda_n;\end{matrix}\ 1,\ldots,1\right)$$

$$=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}n)\Gamma(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}n+\Lambda)}{\Gamma(\frac{1}{2}-\frac{1}{2}n)\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha+\frac{1}{2}n+\Lambda)} \tag{39}$$

where

$$\mathrm{Re}(1-\alpha-n)>0 \qquad \Lambda\equiv\lambda_1+\ldots+\lambda_n.$$

Several additional reduction and summation formulae for the multivariable hypergeometric function (6) can be derived in this manner from the work of Srivastava and Karlsson (1985, §§ 1.3 and 1.4). We choose to leave the details involved as worthwhile exercises for the users of such classes of multiple hypergeometric series as those considered here.

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